QUESTION 1 (15 Marks) **Marks**

(a) By "completing the square" or otherwise, find the minimum value of $2x^2 - 4x + 5$ for all real values of x. **2**

(b) A radioactive substance of mass M decays at a rate proportional to its mass present, ie; $M = M_0 e^{kt}$.
Its initial mass is 400 grams and reduces to 300 grams after 2 years.

(i) How many grams have decayed at the end of four years? **4**

(ii) The half-life of the substance is the time taken for the mass to be reduced by half. What is the half-life of the substance, (to the nearest month). **3**

(c) The equation of motion of an object moving x *cm* about a fixed point O, after t seconds, along a straight line is given by $x = \tan t$ for $0 \le t < \dfrac{\pi}{2}$.

(i) Express its velocity $v(t)$ and acceleration $a(t)$ in terms of t. **3**

(ii) Show that its acceleration is $a(t) = 2x\,(\,1 + x^2\,)$. **3**

QUESTION 2 (15 Marks)

(a) For what values of k is $kx^2 + (3+k)x + (3+k)$ positive definite. **4**

(b) Nine marbles numbered 1, 2, 3, 4, 5, 6, 7, 8 & 9 are placed in a bag and three are drawn out at random (each not replaced). What is the probability that the sum of the numbers on the three marbles drawn is odd. **3**

(c) The equation of motion of a particle moving x metres along a straight line after t seconds is given by $x(t) = 2t^3 - 6t^2 + 3$.

(i) What is the particle's initial speed and acceleration. **2**

(ii) When and where does it first come to rest. **2**

(iii) Sketch a velocity/time graph and briefly explain what is happening to the motion when $t = 1$ second. **2**

(iv) Briefly describe the motion of the particle **2**

QUESTION 3 (15 Marks) **Marks**

(a) The population of two colonies after t years is given by $P_1 = 2000e^{0.138t}$ and $P_2 = 5000e^{0.04t}$. The initial population of each was recorded on the 1st January, 2006.

(i) How long will it take for the population P_1 to triple. (Answer to the nearest year.) **2**

(ii) Calculate the year and month when both populations are the same. **3**

(iii) Calculate the rate at which P_1 is increasing at this time. **2**

(b) Two chess players, Bostik and Spastik, play three games of chess to contest a win. In any game they play, the probability that Bostik wins is $\frac{5}{10}$ and the probability that Spastik wins is $\frac{4}{10}$. What is the probability that Bostik wins the competition. **3**

(c) An object moves x *cm* along a straight line after t seconds with its velocity function $v = -e^{-2t}$ for $t \geq 0$ and is initially at the origin.

(i) Derive an expression for its displacement as a function of time. **3**

(ii) Neatly sketch $x = f(t)$ for $t \geq 0$. **2**

QUESTION 4 (15 Marks)

(a) Find all real values of t for which the quadratic equation $\frac{x^2 - x + 1}{x^2 + x + 1} = t$ has real and different roots. **4**

(b) A particle of unit mass is projected vertically upwards from a point O with a velocity of 25 *m/s* and has an acceleration of $-10 \ m/s^2$.

(i) Find its velocity and height above O after a time $t>0$. **3**

(ii) Find its maximum height of projection. **2**

(b) A deck of cards contains the red Jacks, Queens, Kings and Aces from a normal pack of 52 playing cards (ie; the Hearts and Diamonds). Jenny is dealt two cards from this deck of cards. What is the probability that:

(i) She has two Aces if she announces that she has at least one Ace. **2**

(ii) She has a pair if she announces that she does not have the Ace of Hearts. **2**

(iii) She has two cards of the same suit if she announces that she has at least one Heart and one King. **2**

END of PAPER

YEAR 12 TERM 2
H.S.C. ASSESSMENT 2006
SOLUTIONS

1 (a) $2x^2 - 4x + 5$

$= 2(x^2 - 2x + \frac{5}{2})$

$= 2(x^2 - 2x + 1 + \frac{3}{2})$

$= 2(x-1)^2 + 3$

$\therefore$ MIN of 3 at $x = 1$

OR using $x = -\frac{b}{2a}$

$x = 1 \quad \therefore 2(1)^2 - 4(1) + 5$

$= 3$

(b)(i) $M = M_0 e^{kt}$

$= 400 e^{kt}$

When $t = 2$, $M = 300$

$\therefore 300 = 400 e^{2 \cdot k} \quad \therefore k = \frac{1}{2} \ln \frac{3}{4}$

When $t = 4$

$M = 400 e^{\frac{4}{2} \ln \frac{3}{4}}$

$= 400 \left(\frac{3}{4}\right)^2$

$= 225$

$\therefore$ 175 gms. decayed.

(ii) When $M = \frac{1}{2} M_0$

$\therefore \frac{1}{2} M_0 = M_0 e^{\frac{t}{2} \ln \frac{3}{4}}$

$\therefore \ln \frac{1}{2} = \frac{t}{2} \ln \frac{3}{4}$

$\therefore \quad t = \frac{2 \ln 0.5}{\ln 0.75}$

$\therefore t = 4.82$ years

$= 4$ years 10 months.

(c) (i) $x = \tan t$

$\therefore v(t) = \sec^2 t \quad -(1)$

$\therefore a(t) = 2 \sec t . \sec t \tan t$

$\therefore a(t) = 2 \sec^2 t \tan t \quad -(2)$

(ii) $a(t) = 2 \tan t (1 + \tan^2 t)$

$\therefore a(t) = 2x(1 + x^2)$

since $x = \tan t$

2 (a) $Kx^2 + (3+K)x + (3+K)$

We require $K > 0$ and $\Delta < 0$

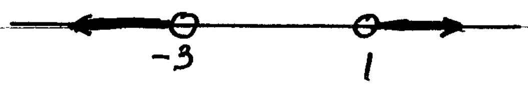
$\Delta = (3+K)^2 - 4K(3+K)$

$= (3+K)(3+K-4K)$

$= (3+K)(3-3K)$

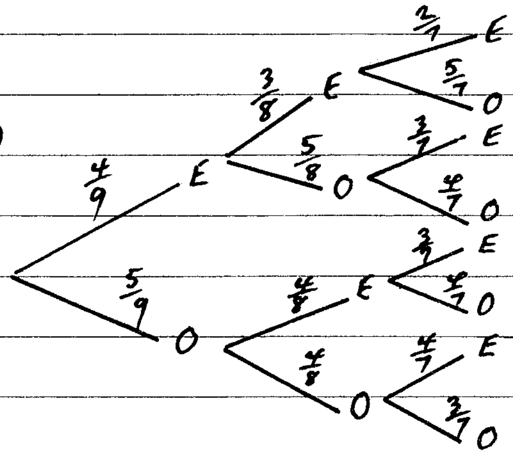
$= 3(3+K)(1-K)$

For $\Delta < 0$



For $K > 0$ we require only $K > 1$.

(b)



$P = P(E+E+O) + P(O+O+O)$

$= 4\left(\frac{4}{9} \times \frac{3}{8} \times \frac{5}{7}\right)$

$= \frac{4 \times 5}{3 \times 2 \times 7}$

$\therefore P = \frac{10}{21}$

2 (c)(i) $x(t) = 2t^3 - 6t^2 + 3$

$v(t) = 6t^2 - 12t$

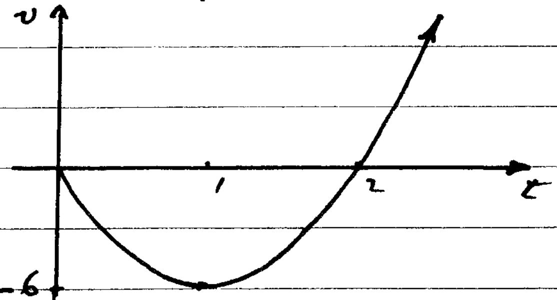
$a(t) = 12t - 12$

When $t = 0$

$v(0) = 0$ m/s and $a(0) = -12$ m/s.

(ii) It comes to rest when $v = 0$ ie; when $t = 0$ or $t = 2$. Since $t > 0$ $\therefore$ when $t = 2$ and $x = -5$ ie; 5 metres left of 0.

(iii)



When $t = 1$ it reaches maximum speed OR zero force acting on the particle

(iv) The particle starts 3 m right of 0 increasing its speed for 1 sec. then slows down and stops 5 m left of 0 and passing through 0 between 2 and 3 seconds. It then increases its speed moving right of 0 indefinitely.

3 (a)(i) $P_1 = 2000e^{0.138t}$ $\quad P_2 = 5000e^{0.04t}$

when $P_1 = 4000$

$4000 = 2000e^{0.138t}$

$\therefore \ln 2 = 0.138t$

$\therefore t = 5.023$

$\therefore t = 5$ years (nearest year)

(ii) when $P_1 = P_2$

$2000e^{0.138t} = 5000e^{0.04t}$

$\therefore e^{0.098t} = \frac{5}{2}$

$\therefore 0.098t = \ln 2.5$

$\therefore t = 9.349$

$= 9$ years 5 months

$\therefore$ During May 2015

(iii) $\frac{dP_1}{dt} = 2000 \times 0.138e^{0.138t}$

$= 276e^{0.138t}$

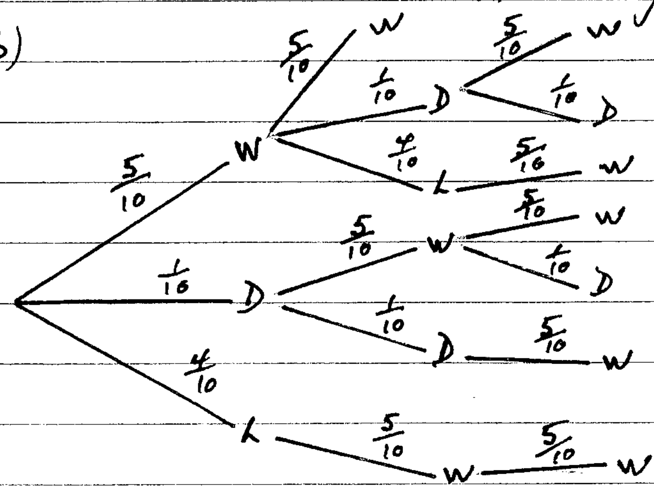
when $t = 9.349$

$\frac{dP_1}{dt} = 276e^{1.290162}$

$= 1002.8$

$\simeq 1003$ /year.

(b)



$P = \left(\frac{5}{10} \times \frac{5}{10}\right) + \left(\frac{5}{10} \times \frac{1}{10} \times \frac{5}{10}\right) + \left(\frac{5}{10} \times \frac{1}{10} \times \frac{1}{10}\right) + \left(\frac{5}{10} \times \frac{4}{10} \times \frac{5}{10}\right)$

$+ \left(\frac{1}{10} \times \frac{5}{10} \times \frac{5}{10}\right) + \left(\frac{1}{10} \times \frac{5}{10} \times \frac{1}{10}\right) + \left(\frac{1}{10} \times \frac{1}{10} \times \frac{5}{10}\right) + \left(\frac{4}{10} \times \frac{5}{10} \times \frac{5}{10}\right)$

$= \frac{25}{100} + \frac{25}{1000} + \frac{5}{1000} + \frac{100}{1000} + \frac{135}{1000}$

$\therefore P = \frac{103}{200}$

3\. (c) (i) $v = \frac{dx}{dt} = -e^{-2t}$

$\therefore x = -\int e^{-2t}\,dt$

$\therefore x = \frac{1}{2} e^{-2t} + C$

When $t = 0$, $x = 0$

$\therefore 0 = \frac{1}{2} + C \quad \therefore C = -\frac{1}{2}$

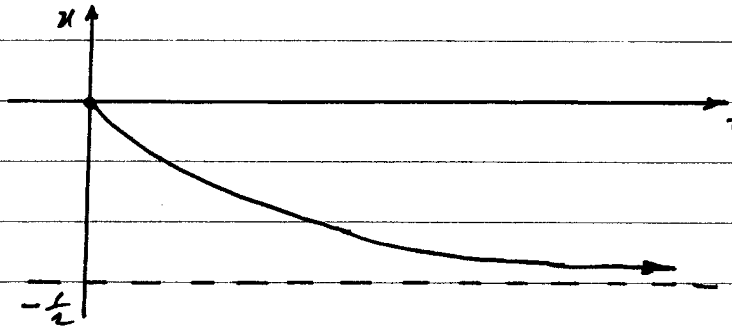
$\therefore x = \frac{1}{2} e^{-2t} - \frac{1}{2}$

$\therefore x = \frac{1}{2}\left(e^{-2t} - 1\right)$

(ii) When $t = 0$, $x = 0$

Let $x = \frac{1}{2}\left(\frac{1}{e^{2t}} - 1\right)$

As $t \to \infty$, $\frac{1}{e^{2t}} \to 0 \quad \therefore t \to -\frac{1}{2}$



4\. (a) $\frac{x^2 - x + 1}{x^2 + x + 1} = t$

$\therefore x^2 - x + 1 = tx^2 + tx + t$

$\therefore (t-1)x^2 + (t+1)x + (t-1) = 0$

For real and different roots, we require $\Delta > 0$

Now $\Delta = (t+1)^2 - 4(t-1)^2$

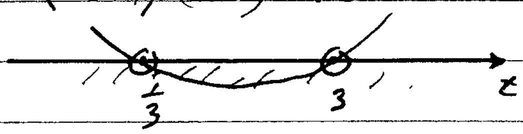
$= t^2 + 2t + 1 - 4t^2 + 8t - 4$

$= -3t^2 + 10t - 3$

$= -3(t^2 - 10t + 3)$

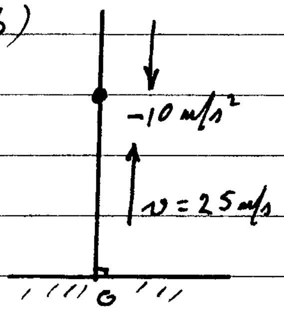
For $\Delta > 0$

$(3t-1)(t-3) < 0$



$\therefore \frac{1}{3} < t < 3$

(b)



$v = \frac{dx}{dt} = 25$

$\therefore x = \int 25\,dt$

$\therefore x = 25t + C$

When $t = 0$, $x = 0 \quad \therefore C = 0$

$\therefore x = 25t$

Now $\frac{dv}{dt} = -10$

$\therefore v = -\int 10\,dt$

$\therefore v = -10t + C$

When $t = 0$, $v = 0 \quad \therefore C = 0$

$\therefore v = -10t$

(i) Now $\underline{v = 25 - 10t}$ (since $v\uparrow = 25\ m/s$)

$\therefore \frac{dx}{dt} = 25 - 10t$

$\therefore x = \int (25 - 10t)\,dt$

$\therefore x = 25t - 5t^2 + C$

When $t = 0$, $x = 0 \quad \therefore C = 0$

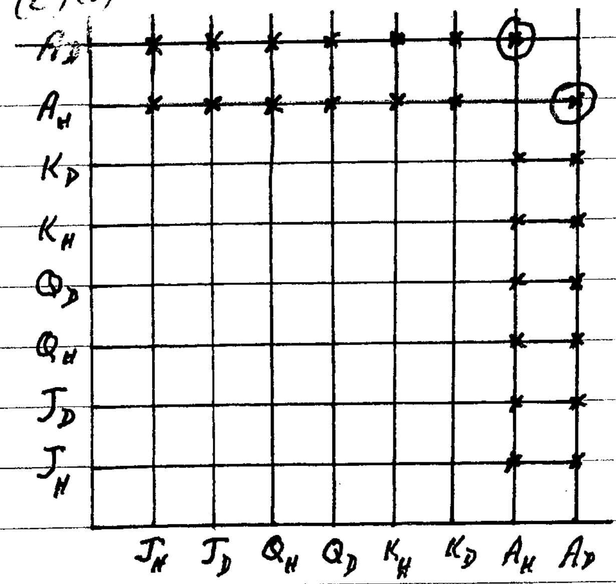
$\therefore \underline{x = 25t - 5t^2}$

(ii) Maximum height when $v = 0$, ie, when $t = \frac{5}{2}$

When $t = \frac{5}{2}$, $x = 25\left(\frac{5}{2}\right) - 5\left(\frac{5}{2}\right)^2$

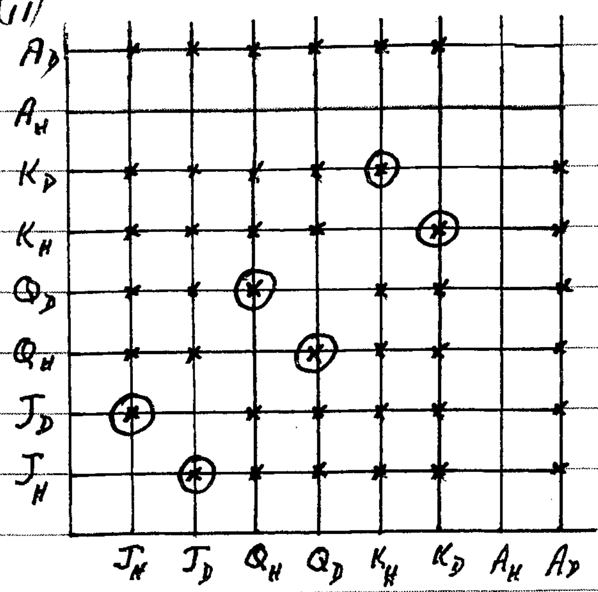
$\therefore x = 31\frac{1}{4}$ m.

4 (c)(i)



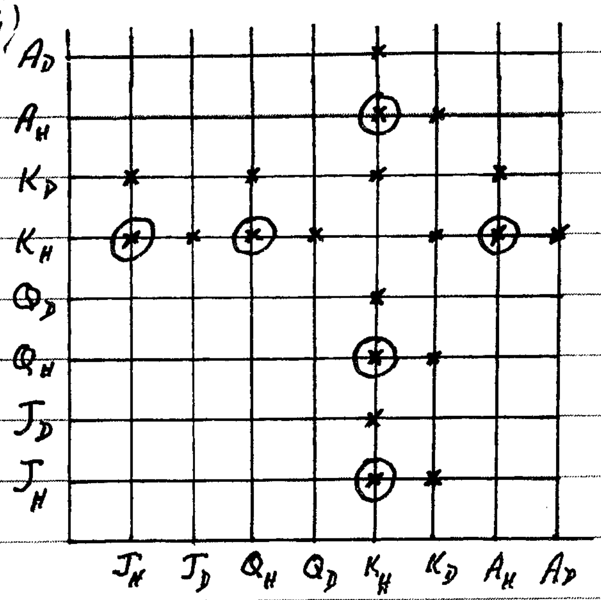
$P = \frac{2}{26} = \frac{1}{13}$

(ii)



$P = \frac{6}{42} = \frac{1}{7}$

(iii)



$P = \frac{6}{20} = \frac{3}{10}$